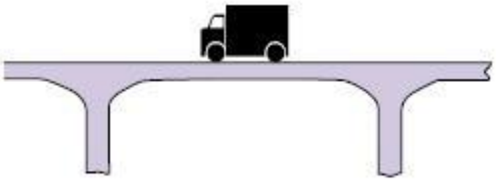
BEAMS and SHAFTS

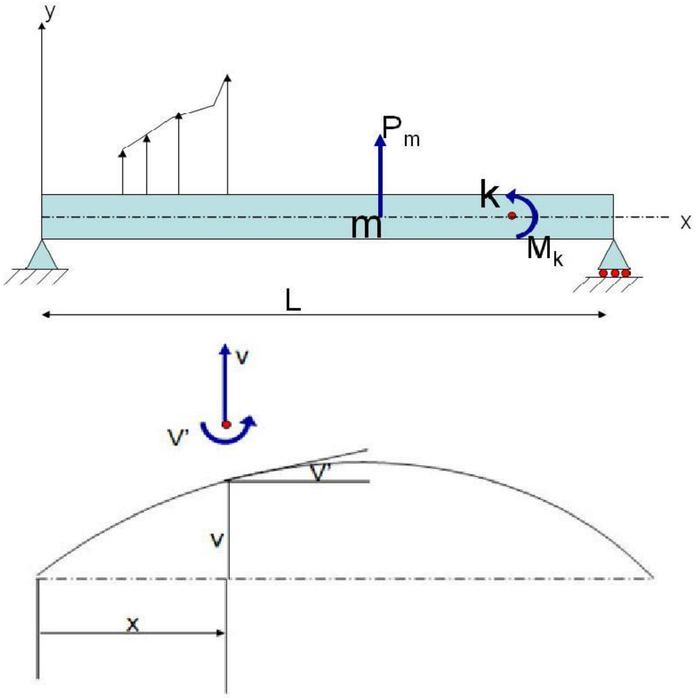
Module-3

Beam element

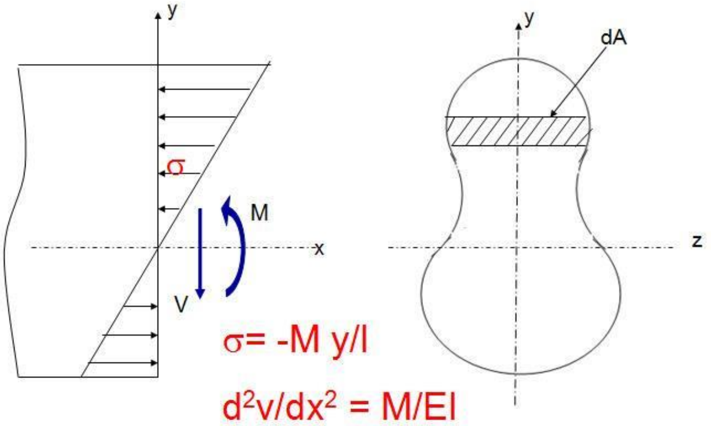
Beam is a structural member which is acted upon by a system of external loads perpendicular to axis which causes bending that is deformation of bar produced by perpendicular load as well as force couples acting in a plane. Beams are the most common type of structural component, particularly in Civil and Mechanical Engineering. A *beam* is a bar-like structural member whose primary function is to support *transverse loading* and carry it to the supports



A truss and a bar undergoes only axial deformation and it is assumed that the entire cross section undergoes the same displacement, but beam on other hand undergoes transverse deflection denoted by v. Fig shows a beam subjected to system of forces and the deformation of the neutral axis



We assume that cross section is doubly symmetric and bending take place in a plane of symmetry. From the strength of materials we observe the distribution of stress as shown.



Where M is bending moment and I is the moment of inertia.
According to the Euler Bernoulli theory. The entire c/s has the same transverse deflection V as the neutral axis, sections originally perpendicular to neutral axis remain plane even after bending

Deflections are small & we assume that rotation of each section is the same as the slope of the deflection curve at that point (dv/dx). Now we can call beam element as simple line segment representing the neutral axis of the beam. To ensure the continuity of deformation at any point, we have to ensure that V & dv/dx are continuous by taking 2 dof @ each node V & θ(dv/dx). If no slope dof then we have only transverse dof. A prescribed value of moment load can readily taken into account with the rotational dof θ .

Potential energy approach

Strain energy in an element for a length dx is given by

$$= \tfrac{1}{2} \int_A \sigma \, \varepsilon \, dA \, dx$$

$$= \tfrac{1}{2} \int_A \sigma \, \sigma/E \, dA \, dx$$

$$= \tfrac{1}{2} \int_A \sigma^2/E \, dA \, dx$$

But we know σ = M y / I substituting this in above equation we get.

$$= \frac{1}{2}\int_A \frac{M^2}{EI^2} y^2 dA\, dx$$

$$= \frac{1}{2}\frac{M^2}{EI^2}\left[\int_A y^2 dA\right] dx$$

$$= \frac{1}{2}\frac{M^2}{EI} dx$$

But

$$M= EI\, d^2v/dx^2$$

Therefore strain energy for an element is given by

$$= \frac{1}{2}\int_0^L EI\,(d^2v/dx^2)^2\, dx$$

Now the potential energy for a beam element can be written as

$$\Pi= \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L p\, v\, dx - \sum_m P_m V_m - \sum_k M_k V'_k$$

P ---- distribution load per unit length

P_m----- point load @ point m

V_m----- deflection @ point m

M_k----- momentum of couple applied at point k

V'_k----- slope @ point k

Hermite shape functions:

1D linear beam element has two end nodes and at each node 2 dof which are denoted as Q_{2i-1} and Q_{2i} at node i. Here Q_{2i-1} represents transverse deflection where as Q_{2i} is slope or rotation. Consider a beam element has node 1 and 2 having dof as shown.

The shape functions of beam element are called as Hermite shape functions as they contain both nodal value and nodal slope which is satisfied by taking polynomial of cubic order

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

that must satisfy the following conditions

ξ	H_1	H_1'	H_2	H_2'	H_3	H_3'	H_4	H_4'
$\xi = -1$	1	0	0	1	0	0	0	0
$\xi = 1$	0	0	0	0	1	0	0	1

Applying these conditions determine values of constants as

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 1

$$H_1 = 1,\quad H_1' = 0\ ,\quad \xi = -1$$

$$1 = a_1 - b_1 + c_1 - d_1 \longrightarrow ①$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 - 2c_1 + 3d_1 \longrightarrow ②$$

$$H_i = a_i + b_i\,\xi + c_i\,\xi^2 + d_i\,\xi^3$$

@ node 2

$$H_1 = 1,\ \ H_1' = 0\ ,\ \ \xi = 1$$

$$0 = a_1 + b_1 + c_1 + d_1 \longrightarrow ③$$

$$H_1' = \frac{dH_1}{d\xi} = 0 = b_1 + 2c_1 + 3d_1 \longrightarrow ④$$

Solving above 4 equations we have the values of constants

$$a_1 = \tfrac{1}{2}\ ,\ b_1 = -\tfrac{3}{4}\ ,\ \ c_1 = 0,\ \ d_1 = \tfrac{1}{4}$$

Therefore

$$H_1 = \tfrac{1}{4}\,(2 - 3\xi + \xi^3)$$

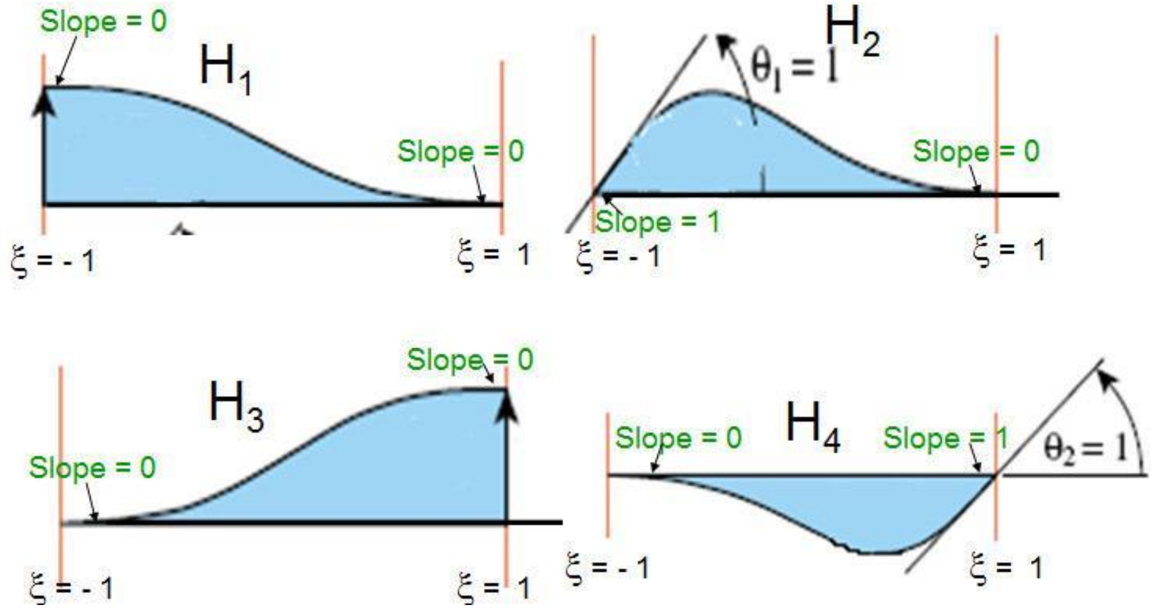
Similarly we can derive

$$H_2 = \tfrac{1}{4}\,(1 - \xi - \xi^2 + \xi^3)$$

$$H_3 = \tfrac{1}{4}\,(2 + 3\xi - \xi^3)$$

$$H_4 = \tfrac{1}{4}\,(-1 - \xi + \xi^2 + \xi^3)$$

Following graph shows the variations of Hermite shape functions



Stiffness matrix:

Once the shape functions are derived we can write the equation of the form

$$V(\xi) = H_1V_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3V_3 + H_4\left(\frac{dv}{d\xi}\right)_2$$

But

$$\frac{dv}{d\xi} = \frac{dv}{dx}\;\frac{dx}{d\xi}$$

$$= \frac{dv}{dx}\frac{Le}{2}$$

ie

$$V(\xi) = H_1V_1 + H_2\left(\frac{dv}{dx}\frac{Le}{2}\right) + H_3V_3 + H_4\left(\frac{dv}{dx}\frac{Le}{2}\right)$$

$$V(\xi) = H_1q_1 + H_2q_2\frac{L_e}{2} + H_3q_3 + H_4q_4\frac{L_e}{2}$$

We know $V = H\,q$

where

$$H = \left(H_1 \quad H_2\frac{L_e}{2} \quad H_3 \quad H_4\frac{L_e}{2}\right)$$

Strain energy in the beam element we have

$$= \tfrac{1}{2}\int_0^L EI\,\underbrace{(d^2v/dx^2)^2}\,dx$$

$$\frac{d^2v}{dx^2} = \frac{d}{dx}\left(\frac{dv}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{2}{L_e}\frac{dv}{d\xi}\right)$$

$$= \frac{2}{L_e}\frac{d}{dx}\left(\frac{dv}{d\xi}\right)$$

$$= \frac{2}{L_e}\frac{d}{dx}(m) \qquad \text{Where } m = \frac{dv}{d\xi}$$

$$= \frac{2}{L_e}\left(\frac{2}{L_e}\frac{dm}{d\xi}\right)$$

$$\frac{d^2v}{dx^2} = \frac{4}{L_e^2}\left(\frac{d^2v}{d\xi^2}\right) \qquad \left(\frac{d^2v}{dx^2}\right)^2 = \frac{16}{L_e^4}\left(\frac{d^2v}{d\xi^2}\right)^2$$

$$V = H\,q$$

$$\left(\frac{d^2v}{dx^2}\right)^2 = q^T\,\frac{16}{L_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right) q$$

Where

$$\left(\frac{d^2H}{d\xi^2}\right) = \left[\frac{3\xi}{2}\ ,\ \left(\frac{-1+3\xi}{2}\right)\frac{l_e}{2}\ ,\ \frac{-3\xi}{2}\ ,\ \left(\frac{1+3\xi}{2}\right)\frac{l_e}{2}\right]$$

Therefore total strain energy in a beam is

$$= \tfrac{1}{2}\int_e EI\,(d^2v/dx^2)^2\,dx$$

$$= \tfrac{1}{2}\int_e EI\,(d^2v/dx^2)^2\,l_e/2\ d\xi$$

$$= \frac{EI}{2}\,\frac{L_e}{2}\int_e q^T\,\frac{16}{L_e^4}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right) q\ d\xi$$

$$= \tfrac{1}{2}\,q^T\boxed{\left\{\frac{8EI}{L_e^3}\int_{-1}^{+1}\left(\frac{d^2H}{d\xi^2}\right)^T\left(\frac{d^2H}{d\xi^2}\right)\right\}}\,q\,d\xi$$

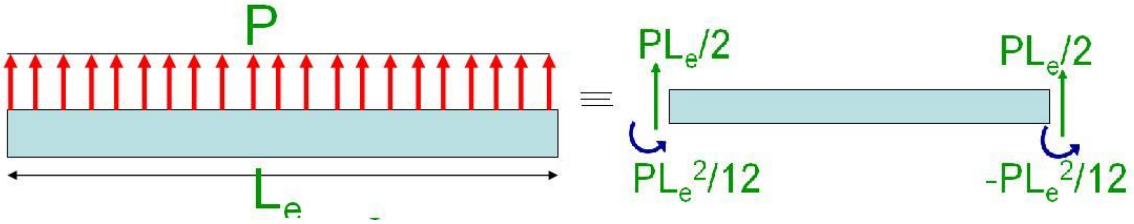
$$= \tfrac{1}{2}\,q^T\ K\ q$$

Now taking the K component and integrating for limits -1 to +1 we get

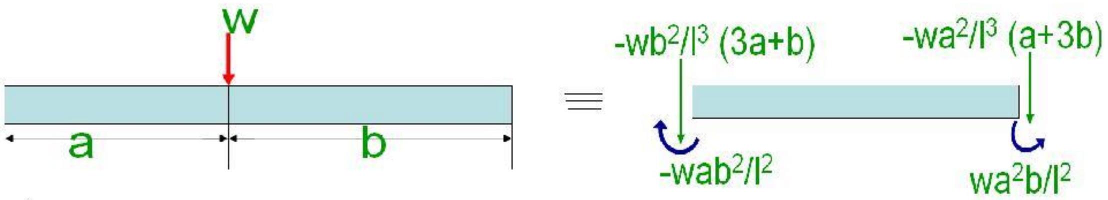
$$K = \frac{EI}{Le^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

Beam element forces with its equivalent loads

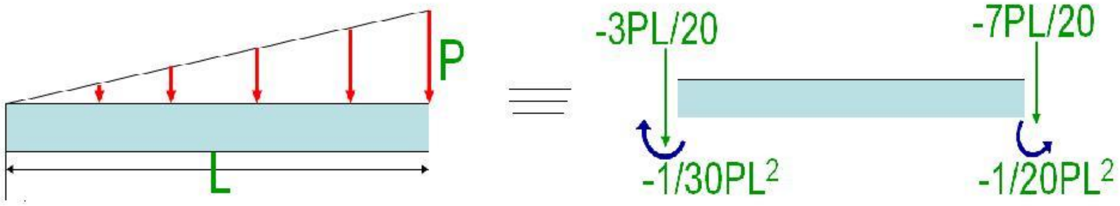
Uniformly distributed load



Point load on the element



Varying load



Bending moment and shear force

We know

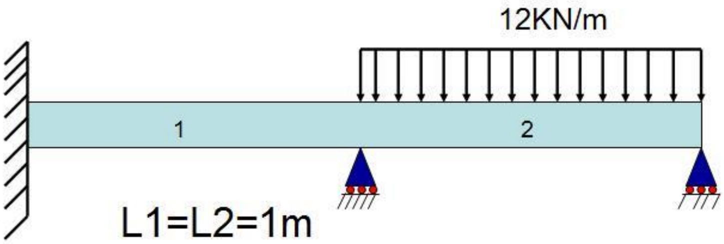
$$M=EI\left(\frac{d^2v}{dx^2}\right) \quad V=\left(\frac{dM}{dx}\right) \quad V=Hq$$

Using these relations we have

$$M=\frac{EI}{l_e^2}\left[6\xi q_1+(3\xi-1)l_e q_2-6\xi q_3+(3\xi+1)l_e q_4\right]$$

$$V=\frac{6EI}{l_e^3}\left[2q_1+l_e q_2-2q_3+l_e q_4\right]$$

Example 8



L1=L2=1m

E = 200GPa

I = 4 X 10^6N/mm^4

Solution:

Let's model the given system as 2 elements 3 nodes finite element model each node having 2 dof. For each element determine stiffness matrix.

$$K_1 = 8 \text{ X } 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left(\begin{array}{cccc} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \quad K_2 = 8 \text{ X } 10^5 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccc} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{array}\right) \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Global stiffness matrix

$$K = 8 \text{ X } 10^5 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{cccccc} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{array}\right) \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

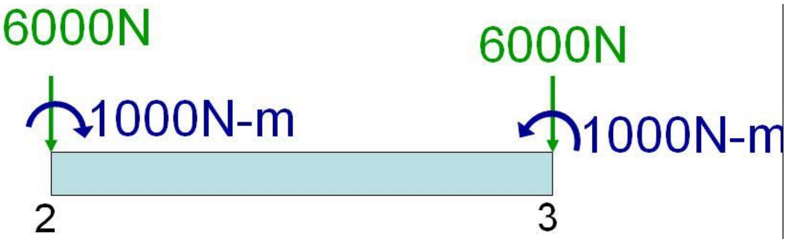
Load vector because of UDL

Element 1 do not contain any UDL hence all the force term for element 1 will be zero.

ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For element 2 that has UDL its equivalent load and moment are represented as



ie

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

From KQ=F we write

$$8 \times 10^5 \begin{pmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

At node 1 since its fixed both q1=q2=0

node 2 because of roller q3=0

node 3 again roller ie q5= 0

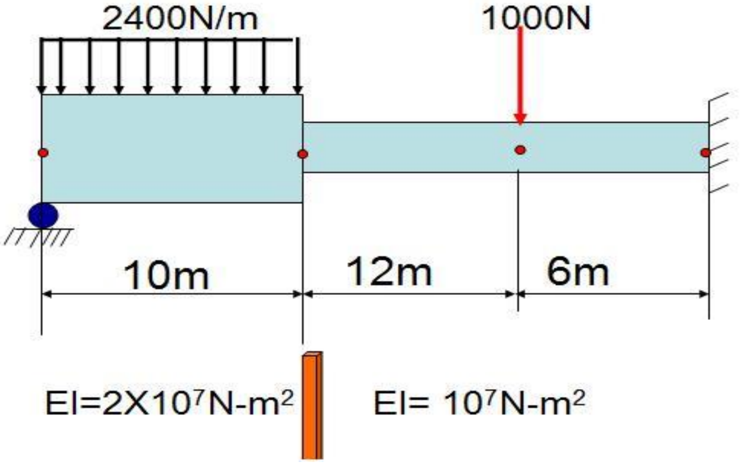
By elimination method the matrix reduces to 2 X 2 solving this we have Q4= -2.679×10^{-4}mm and Q6 = 4.464×10^{-4}mm

To determine the deflection at the middle of element 2 we can write the displacement function as

$$V(\xi) = H_1 q_3 + H_2 q_4 \frac{L_e}{2} + H_3 q_5 + H_4 q_6 \frac{L_e}{2}$$

$$= -0.089\text{mm}$$

Example 9



Solution: Let’s model the given system as 3 elements 4 nodes finite element model each node having 2 dof. For each element determine stiffness matrix. Q1, Q2……Q8 be nodal displacements for the entire system and F1……F8 be nodal forces.

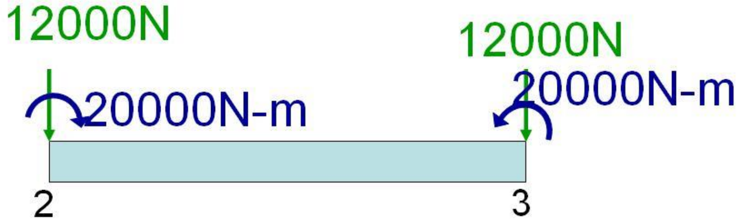
$$K_1 = \frac{2X\,10^7}{10^3}\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline 12 & 60 & -12 & 60 & 1 \\ 60 & 400 & -60 & 200 & 2 \\ -12 & -60 & 12 & -60 & 3 \\ 60 & 200 & -60 & 400 & 4 \end{array} \qquad K_2 = \frac{10^7}{12^3}\begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ \hline 12 & 72 & -12 & 72 & 3 \\ 72 & 576 & -72 & 288 & 4 \\ -12 & -72 & 12 & -72 & 5 \\ 72 & 288 & -72 & 576 & 6 \end{array}$$

$$K_3 = \frac{10^7}{6^3}\begin{array}{cccc|c} 5 & 6 & 7 & 8 & \\ \hline 12 & 36 & -12 & 36 & 5 \\ 36 & 14 & -36 & 72 & 6 \\ -12 & -36 & 12 & -36 & 7 \\ 36 & 72 & -36 & 144 & 8 \end{array}$$

Global stiffness matrix:

$$K = \begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline & & & & & & & & 1 \\ & & & & & & & & 2 \\ & & & & & & & & 3 \\ \multicolumn{8}{c|}{8 \text{ X } 8} & 4 \\ & & & & & & & & 5 \\ & & & & & & & & 6 \\ & & & & & & & & 7 \\ & & & & & & & & 8 \end{array}$$

Load vector because of UDL:
For element 1 that is subjected to UDL we have load vector as



ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} -12000 \\ -20000 \\ -12000 \\ 20000 \end{pmatrix}$$

Element 2 and 3 does not contain UDL hence

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad F_3 = \begin{pmatrix} F5 \\ F6 \\ F7 \\ F8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

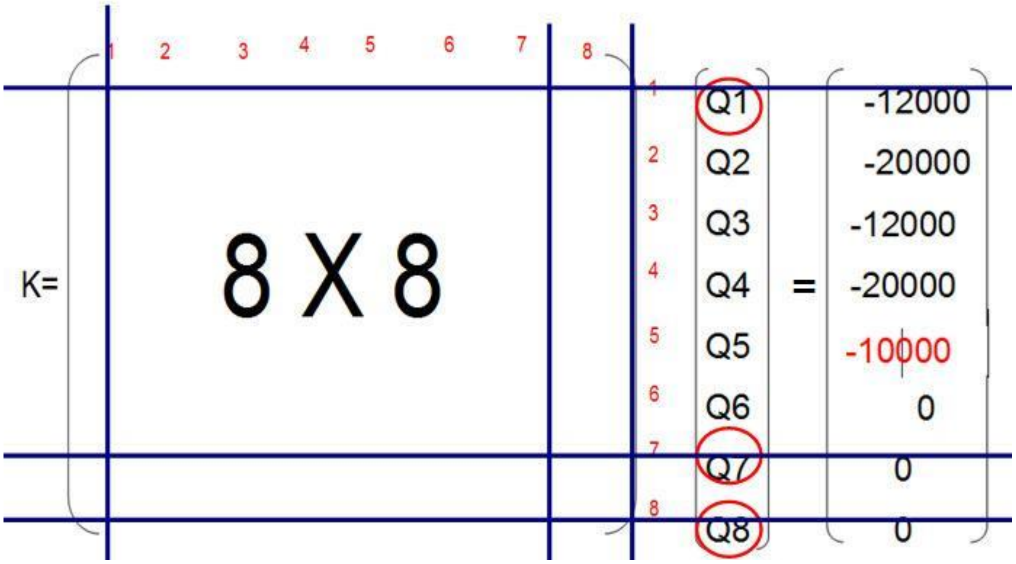
Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \\ F7 \\ F8 \end{pmatrix} = \begin{pmatrix} -12000 \\ -20000 \\ -12000 \\ -20000 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And also we have external point load applied at node 3, it gets added to F5 term with negative sign since it is acting downwards. Now F becomes,

$$F=\begin{Bmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \\ F7 \\ F8 \end{Bmatrix} = \begin{Bmatrix} -12000 \\ -20000 \\ -12000 \\ -20000 \\ 0 \;\; -10000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

From KQ=F



At node 1 because of roller support
q1=0 Node 4 since fixed q7=q8=0
After applying elimination and solving the matrix we determine the values of q2, q3, q4, q5 and q6.